Wigner Functions

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Abstract

Visualising a detailed quantum state is often a very challenging task due to the major differences from the classical world seen around us. Wigner functions are a neat aid to this process and provide a quasi-probability distribution in phase space for a pair of variables. This report calculates the Wigner function for a variety of quantum harmonic oscillator states. Pure coherent and number states are considered along with mixed states and Schrödinger cat states. Finally Wigner functions for thermal states of a quantum harmonic oscillator are calculated for different temperatures.

It is found that coherent states are of the lowest uncertainty and, in the general case, also evolve in time as a classical oscillator would. Number, mixed and Schrödinger cat states are found to be highly quantum states often with no classical analogy while thermal states are found to have an uncertainty that varies with temperature.

Contents

1 Introduction

1.1 Aims

This project aimed to calculate and plot the Wigner function for a variety of pure and mixed states of the quantum harmonic oscillator. Using these plots this report aims to discuss the differences between different quantum states and how they can be visualised or compared to classical states with which we are more familiar.

1.2 Background

Modern quantum mechanics continues to offer an ever more detailed description of the quantum states of yet more complex systems. The deeper and more interesting these descriptions become, the harder it can be to visualise these states, mainly due to their lack of similarities with the much more familiar classical world.

In the classical world a particle has a known position and a momentum, so for an ensemble of particles we can generate a probability distribution, allowing us to know the probability of finding a particle with a given position and momentum. As we know, this is not the case in the Quantum world; the Heisenberg Uncertainty principle stipulates that we cannot know the position and momentum of a particle simultaneously where their uncertainties shrink below a certain level. In this quantum world we must consider a quasi-probability distribution in order to take account of the non-classical outcomes.

In this report we will consider the *Wigner function*, a widely used quasi-probabalistic function with applications ranging from molecular dynamics to quantum optics and much further.

1.2.1 Visualising Quanta

One way of visualising a state is to consider the probability density functions of individual properties of that state. For some variables this is straight forward: if the wavefunction of a state is known to be $\psi(x)$ one can find the probability density in position space by simply squaring it's modulus, $|\psi(x)|^2$.

It is quite straight forward to interpret this idea of position by plotting the probability density and looking at the most likely position to find an object. The *expectation value* for a property is based on this exact feature [4, p.92,eqs:4.31,4.32]

$$\langle x \rangle = \int \psi^*(x)\, \hat{x} \psi(x)\, dx = \langle x \mid \hat{x} \mid x \rangle \tag{1}$$

Where $\hat{x}$ is the operator associated with position x and integrals are over all space.

For other variables however things are not so straight forward. The momentum distribution $\Phi(p)$ can be found as below [1, p.937]

$$\Phi(p) = \frac{1}{A} \int \exp(-ixp/\hbar)\, dx = \langle p \mid \psi \rangle \tag{2}$$

Where p is momentum, A is a constant and, as before, integrals are over all space.

Again, squaring the modulus gives the probability distribution, however visualising this function is not easy without position information. This is where the Wigner function is of use as it allows us to simultaneously plot the momentum and position probability distributions in phase space. The result can help visualise the information in a way with which we are much more comfortable.

1.2.2 The Wigner Function

Plotting in phase space is very useful when it comes to making comparisons between the quantum world and the classical world. As previously mentioned, in the classical world the position and momentum of a particle can be known simultaneously, meaning it is possible to plot its trajectory through phase space. If an ensemble of trajectories are considered a probability distribution in phase space can be plotted, which can conveniently be compared to the Wigner function. This provides a nice way of comparing quantum and classical mechanics. Although the Wigner function can be used for any pair of variables, throughout this report position, x, and momentum, p, will be considered so that comparisons with the classical world can be made. When making these comparisons it is important to remember that the Wigner function is a quasi-probability; it may produce a negative value, for which there is no classical analogy (see the *Fock States* where $n > 0$ for examples, in section 3.4).

The Wigner function is defined as the following [2, pg.64]

$$W\left(q,p\right)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\psi^{*}\left(q-\frac{x}{2}\right)\psi\left(q+\frac{x}{2}\right)e^{ipx/\hbar}dx \tag{3}$$

Or, most generally in dirac notation (making use of the density operator, $\hat{\rho}$):

$$W\left(q,p\right)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\langle q-\frac{x}{2}\mid\hat{\rho}\mid q+\frac{x}{2}\rangle e^{ipx/\hbar}dx \tag{4}$$

1.2.3 The Harmonic Oscillator

Throughout this project the harmonic oscillator has been studied as it is the simplest quantum system. Its wavefunction is [4, p.95,eq:4.69]

$$\psi_n\left(x\right)=\frac{\mathrm{H}_n\left(\frac{x}{a}\right)e^{-x^2/2a}}{\left(n!\,2^n a\sqrt{\pi}\right)^{1/2}} \tag{5}$$

Or, in the ground state (as used in section 3.1 and the derrivation in appendix C.1):

$$\psi_0\left(x\right)=\left(\frac{1}{a\sqrt{\pi}}\right)^{1/2}e^{-x^2/2a^2} \tag{6}$$

Where $a=\sqrt{\frac{\hbar}{m\omega}}$

The coherent state of the harmonic oscillator can be found to be the following:

$$\psi_\alpha\left(q\right)=\left(\frac{1}{a\sqrt{\pi}}\right)^{1/2}e^{\left(\alpha^2-|\alpha|^2\right)/2}e^{-\left(\frac{q}{a}-\sqrt{2}\alpha\right)/2} \tag{7}$$

As coherent states are the superposition of every number state, each with a specific probability – the equation above describes pure coherent states, which minimal uncertainty in both their position and momentum.

For thermal states the density operator $\hat{\rho}$ is

$$\hat{\rho}=\sum_{n=0}^{\infty}P_n\mid\alpha\rangle\langle\alpha\mid \tag{8}$$

Where

$$P_n=\frac{\exp\left(\left(n+\frac{1}{2}\right)-\hbar\omega/k_BT\right)}{Z} \tag{9}$$

and

$$Z=\sum_{n=0}^{\infty}P_n\exp\left(\left(n+\frac{1}{2}\right)\frac{\hbar\omega}{k_BT}\right) \tag{10}$$

Therefore thermal states are the superposition of all the number states multiplied by the probability of the particle being found in that state.

1.2.4 Quantum States

The states used in this report have been chosen for their unique attributes.

- The ground state (the Fock state with $n=0$) represents the lowest energy possible from a quantum state. Its shape should be similar to the gaussuan distribution, given below [4, p.58,eq:2.552], where subscript α shows that the term is the sum of its contents over each dimension required, μ is the mean and σ the variance.

$$P=\frac{1}{\sigma_\alpha\sqrt{2\pi}}\exp\left(\frac{-\left(x-\mu\right)_\alpha^2}{\omega\sigma^2}\right) \tag{11}$$

- Number states (or Fock states) are pure states, so called because they cannot be recreated by the combination of other states. In a manner similar to Fourier Transforms, the Fock states can be used as the basis to create other states.

- A Coherent state is the superposition of every number state, each with a specific probability, with the aim of creating the state of the lowest uncertainty at the given energy. The higher energy coherent states should look identical, ie. of the same shape, however they will necessarily be displaced in phase-space because of their additional energy.
- Time dependence in quantum states should bring about a rotation in phase space; just as an oscillator in one dimension plots a sinusoidal path and a two dimensional oscillator plots a circle, the path of the one dimensional oscillator plotted in phase space should carve out a circular 'orbit' around the origin. For states like the ground state where there is complete circular symmetry there should be no evolution in time – the ground state of an oscillator is a non-moving object, therefore it has no time component.
- In Schrödinger's famous thought experiment a cat is placed in a closed box with some poisonous gas. It is then argued that it cannot be known if the cat is alive or dead until the box is opened suggesting that the cat is both alive and dead at the same time. It is therefore expected that the Wigner function for a Schrödinger cat state will be a distribution giving the possibility of being in either one of two clear states or a combination of both.
- Mixed states should return a plot that gives the possibilities of being in clear states but with no information as to which state a particle would be in before a measurement is taken. It is therefore expected that the plot is generally flat with a number of small Gaussian peaks equal to the number of possible states that a particle could be in.
- Increasing the temperature of a harmonic oscillator introduces larger probabilities of a particle being found outside the ground state due to it having extra energy. The Wigner function for the harmonic oscillator should therefore also be a Gaussian but with the standard deviation increasing with temperature.

1.3 Layout

Section 2 of this report discusses the methods used in calculating the Wigner function for a given state. Both numerical and analytical approaches are discusses for a variety of states. In section 3 the resulting Wigner function plots are displayed and there are discussions of how changes to variables related to each state change the Wigner function as a whole. The report then moves on to section 4 where the plots are compared and contrasted with one another and the meanings of the plots are evaluated. Finally section 5 concludes the findings of the project.

2 Method

2.1 Numerical Method

2.1.1 Principles

The MATLAB program that generates the data and the plot from the integrals in section C simply iterates over values for q_0 and p_0, finding a corresponding value for $W_0(q_0, p_0)$ for each, then produces a 3-Dimensional surface plot of the results.

Important parts of the program (see section B.1) are the `quad` functions [3], on lines 40 to 56. These are programatic methods that evaluate the integral of a function between given limits, adaptively, by approximating using Simpson's rule [4, p.61,eq:2.586].

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6} \cdot \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \tag{12}$$

This method is considerably more accurate than simply evaluating the function at a range of points and summing them as, if the method is allowed to assume that the function tends to 0 near the limits, it can watch for additional terms becoming negligible and end the summation early. This means a much smaller δx – the distance between evaluated points – can be used, without incurring massive computational times. MATLAB's `quad` method uses $\delta x = 1 \times 10^6$ as standard, which gives reasonable results in a reasonable timeframe, thus this project uses the default value for δx.

There is a major stumbling block to this integration, if it is attempted numerically. Because there are values of $e^{-1/\hbar^2}$ inside the integration there will be very significant rounding errors. The exponent of very large negative numbers tends to zero and, due to the nature of storing numbers digitally (namely the number of bytes devoted to storing a float – a topic outside the scope of this report), very inaccurate integrals will be produced. In order to counter this dimensional substitution is enacted. If the values of q and p given to a Wigner function are given in specific multiples of $\hbar$ then this problem can be avoided. This process, as it is employed here, is documented in appendix C.1.1 for the ground state solution and used in all the calculations before plotting.

An important choice is the limits for the numerical integration. Too wide and the calculation will be prohibitavely time consuming, too narrow and important data will not be included rendering the calculation unacceptably inaccurate. Because this project deals with the harmonic oscillator there is always an $e^{-x^2/\hbar}$ term inside the integral (see eq.6) which allows the assumption that $x \approx \sqrt{\hbar}$, in order for there to be a significant contribution to the integral. Because of the dimensional substitution mentioned above it is reasonable to assume that there will be no significant contribution to the integral outside the range $-10^{-3} < x_0 < 10^3$, thus defining the integration limits used.

2.1.2 Calculation of Individual States

Each of the equations used in the numerical calculations can be found in appendix C. The MATLAB code for each of these equations is separated into the integral's multiplicant, which can be found on the relevant line in `wigner.m`, in appendix B.1, and the function inside the integral, which can be found in the appropriate subsection of appendix B.

2.2 Analytical Method

An analytical solution was found for the ground state, the coherent state and for the time dependent coherent states. These are the states where the integral in the Wigner function is simple enough to make their analytical solution simple . The Derrivations appendix (appendix C) holds the working completed to find each of the analytical solutions for the states mentioned above.

An analytical solution for the Wigner function for the Schrödinger Cat states is simple enough in principle – but, as explained in appendix C.1.1, there are 4 separable components to the integral; two are coherent states (and thus an analytical solution is ready from appendix C.2) but the other two, while solvable, are long winded solutions that offer little benefit over the numerical solutions used below.

3 Results

3.1 Ground State

The Wigner function plots for the ground state of a quantum harmonic oscillator are shown for an analytical solution in figure 1 and a numerical solution in figure 2. It can be seen that the results do not differ and return a Gaussian surface centred on $p = q = 0$.

3.1.1 Analytical Solution

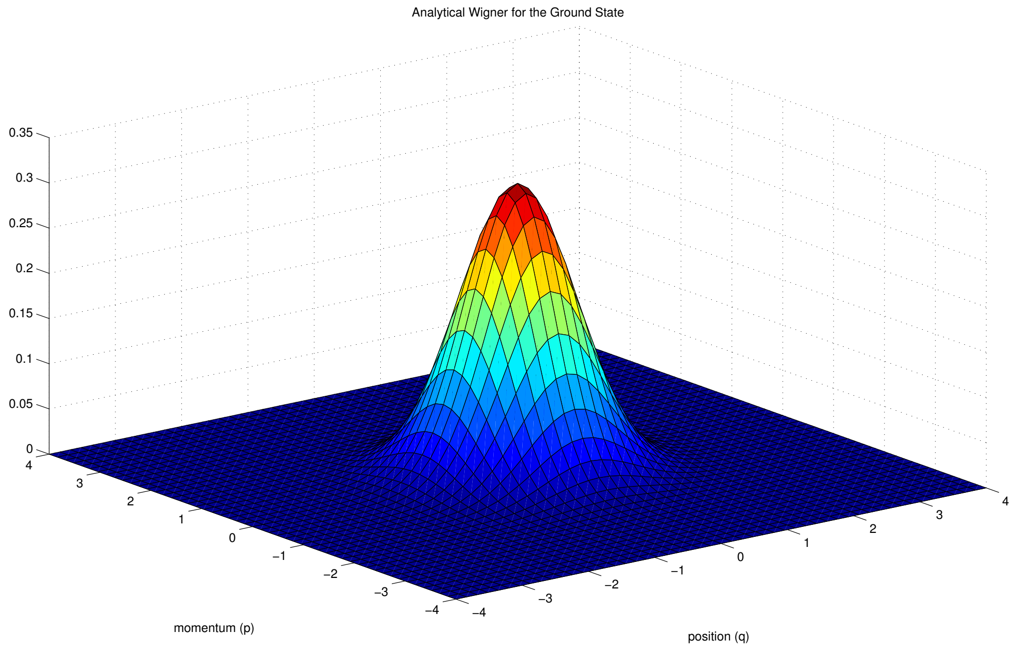

Figure 1: An analytically calculated Wigner function for the ground state of the harmonic oscillator

3.1.2 Numerical Solution

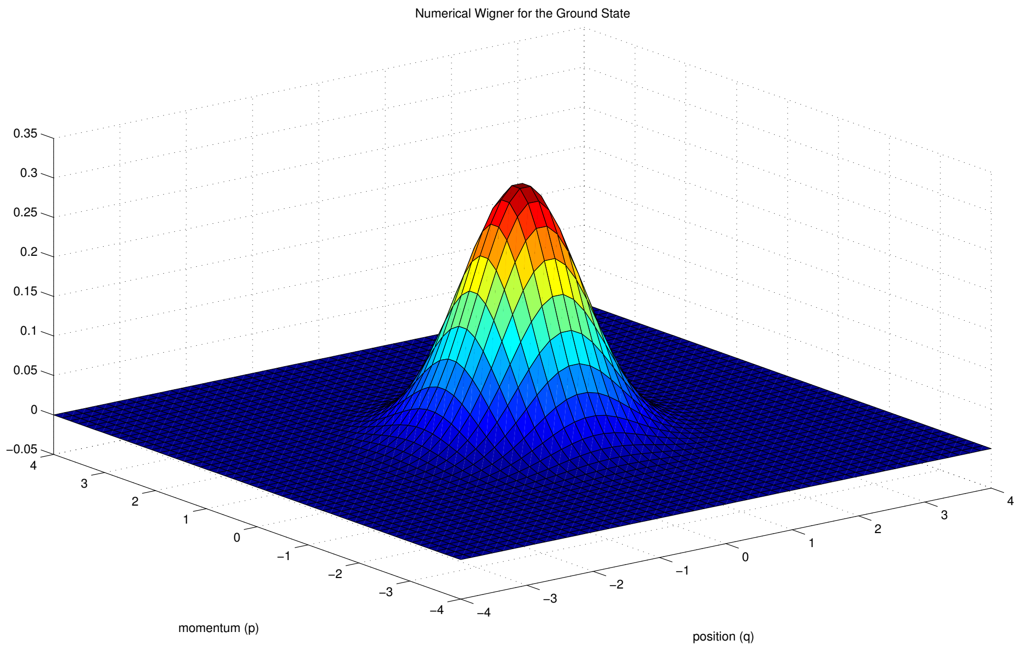

Figure 2: A numerically calculated Wigner function for the ground state of the harmonic oscillator

3.2 Coherent States

Figures 3 and 4 show the Wigner function for a pure coherent state, $\mid \alpha\rangle$, where $\alpha = 0$. The plot is very similar to that for the ground state of the harmonic oscillator but with a shaper, narrower Gaussian function.

Generally for a coherent state α will be a complex number. The Wigner function for $\alpha = (1\text{–}i)$ is plotted for an analytical solution in figure 5, whilst figure 6 shows a numerically calculated solution for the same value of α. The Wigner function has exactly the same Gaussian form in both cases as expected. It can be seen that the Gaussian shape is identical for that of the $\alpha = 0$ plot, but translated to be centred around $p = +\sqrt{2}$, $q = -\sqrt{2}$. Using different values of α will always yield the same Gaussian shape but translated to be centred around $p = +\sqrt{2}\Im\alpha$, $q = +\sqrt{2}\Re\alpha$.

3.2.1 Analytical Solution ($\alpha = 0$)

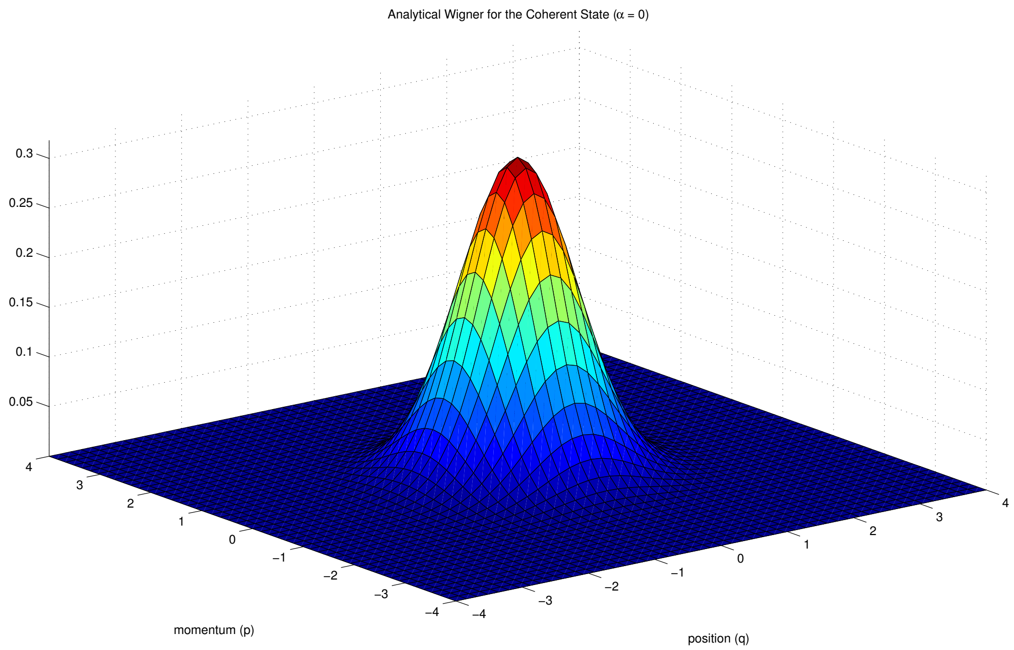

Figure 3: An analytically calculated Wigner function for the coherent state of the harmonic oscillator where $\alpha = 0$

3.2.2 Numerical Solution ($\alpha = 0$)

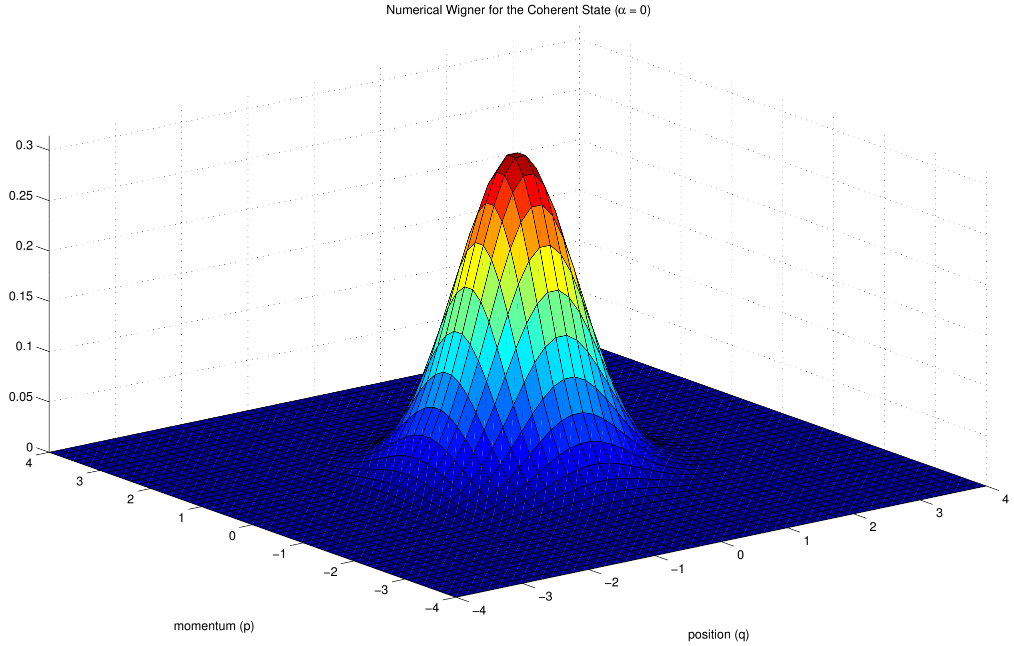

Figure 4: A numerically calculated Wigner function for the coherent state of the harmonic oscillator where $\alpha = 0$

3.2.3 Analytical Solution ($\alpha = 1 - i$)

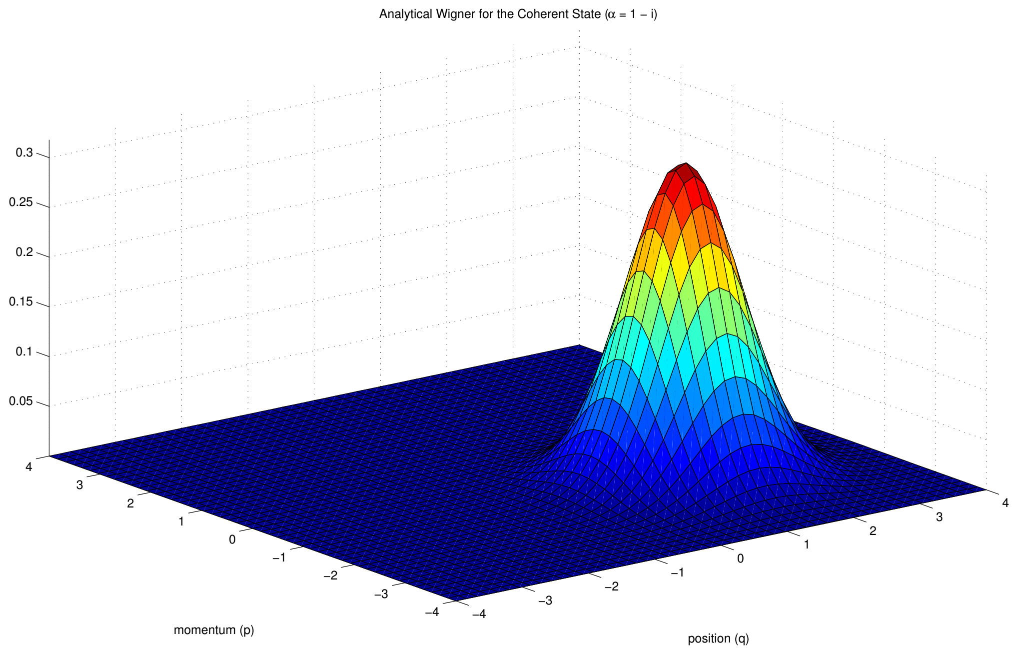

Figure 5: An analytically calculated Wigner function for the coherent state of the harmonic oscillator where $\alpha = 1 - i$

3.2.4 Numerical Solution ($\alpha = 1 - i$)

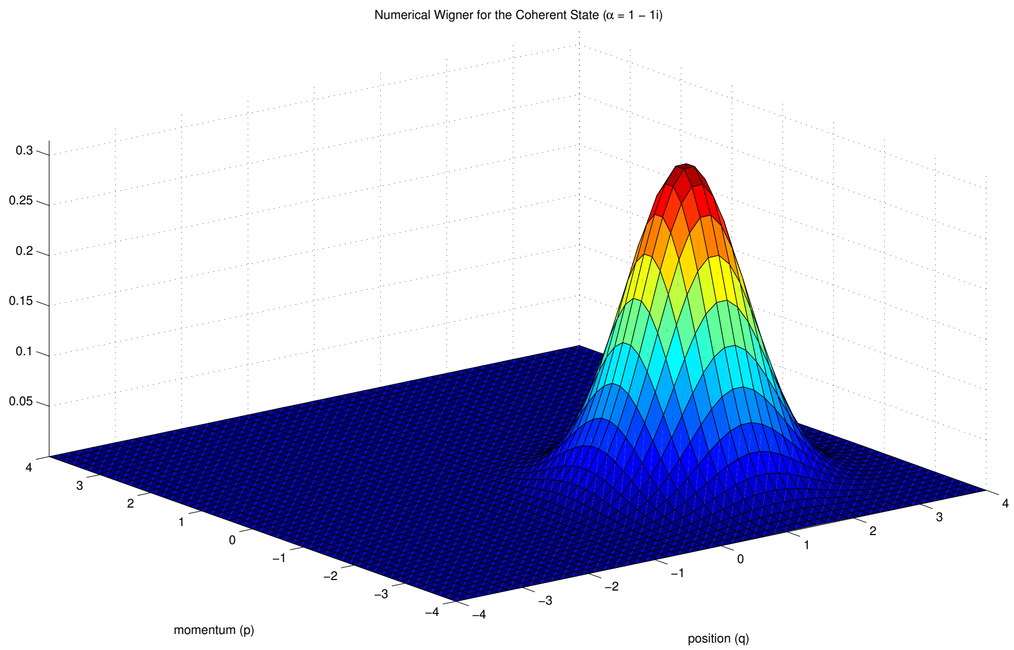

Figure 6: A numerically calculated Wigner function for the coherent state of the harmonic oscillator where $\alpha = 1 - i$

3.3 Time Dependant Coherent States

The ability to give the oscillator a position and momentum at a given time leads nicely to considering the time evolution of the Wigner function. Figure 7 shows the coherent state for $\alpha = (1 - i)$ at four separate times and figure 8 shows the same plots but viewed from above. A clear circular orbit centred on $p = q = 0$ can be observed as would be observed for a classical oscillator. Altering the value of α leads to different stating positions, but a circular orbit around $p = q = 0$ will always proceed with the shape of the Gaussian never altering.

For each of these plots, $\alpha = 1 - i$

3.3.1 Side-View Analytical Solution $(0 < t < \pi/2)$

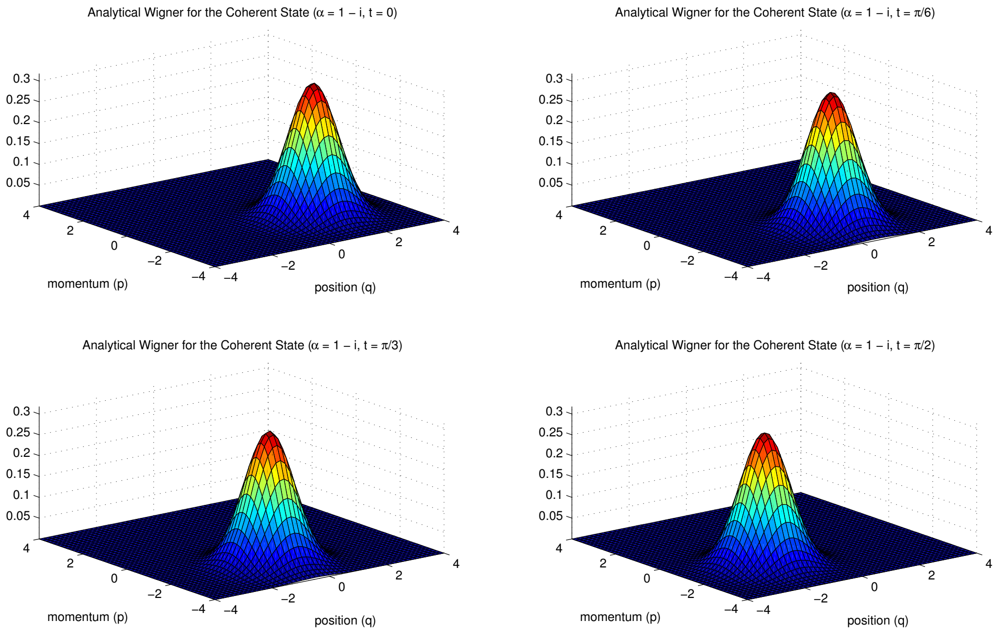

Figure 7: The time dependence of a coherence state

3.3.2 *Bird's Eye* View Analytical Solution ($0 < t < \pi/2$)

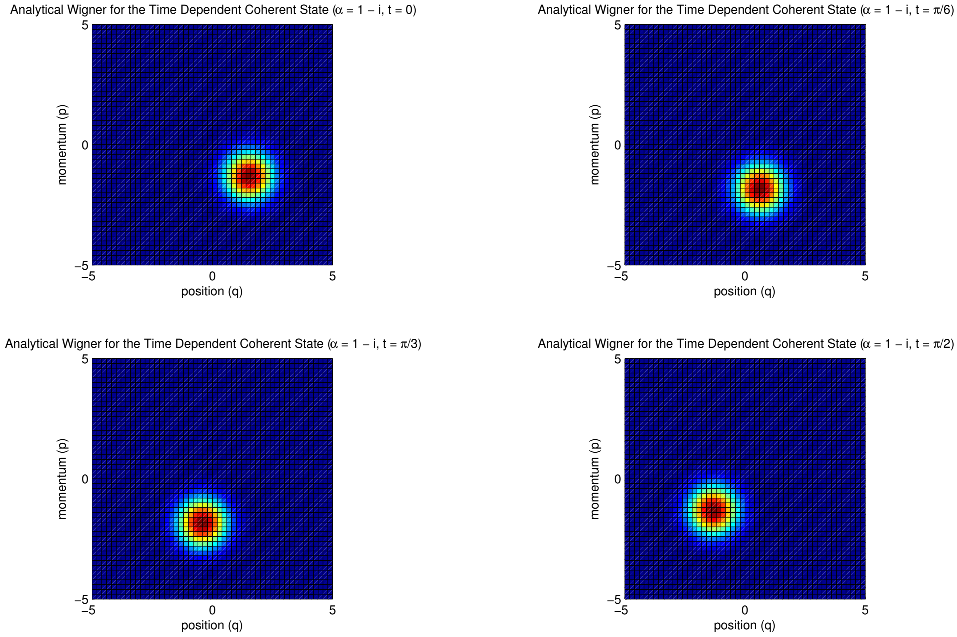

Figure 8: The same time dependence of a coherence state, viewed from above

3.4 Fock/Number States

The number states make for an interesting comparison to coherent states as they are generally very non-classical. The following figures show the number states $0 \leq n \leq 4$. Each Wigner function has $n+1$ turning points in the plot and ripples away from the $q = p = 0$ point as water would ripple away from the point of impact if a stone were dropped into a pond. The plots always have circular symmetrical about $p = q = 0$ and as a result no changes in the shape of the plot would be seen if a time dependence was introduced.

3.4.1 Numerical Solution ($n = 0$)

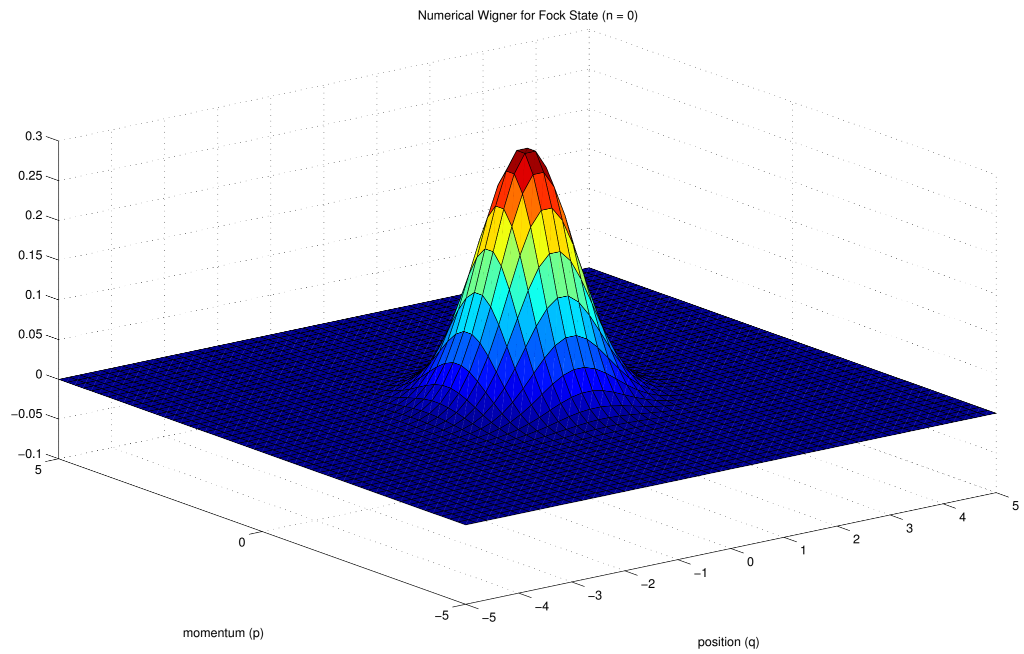

Figure 9: The Fock state for $n = 0$

3.4.2 Numerical Solution ($n = 1$)

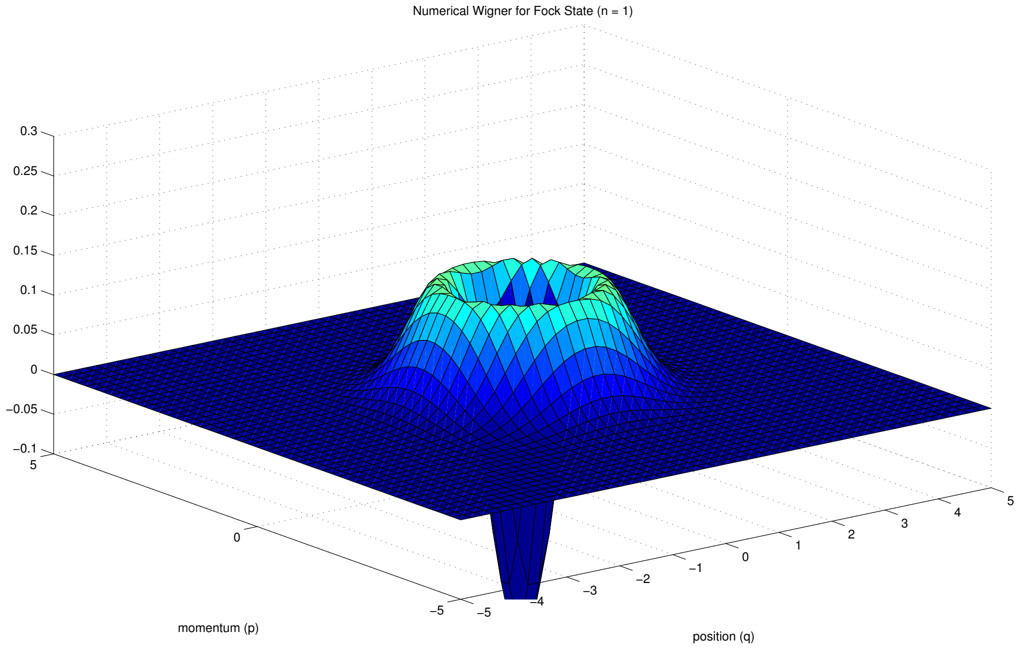

Figure 10: The Fock state for $n = 1$

3.4.3 Numerical Solution ($n = 2$)

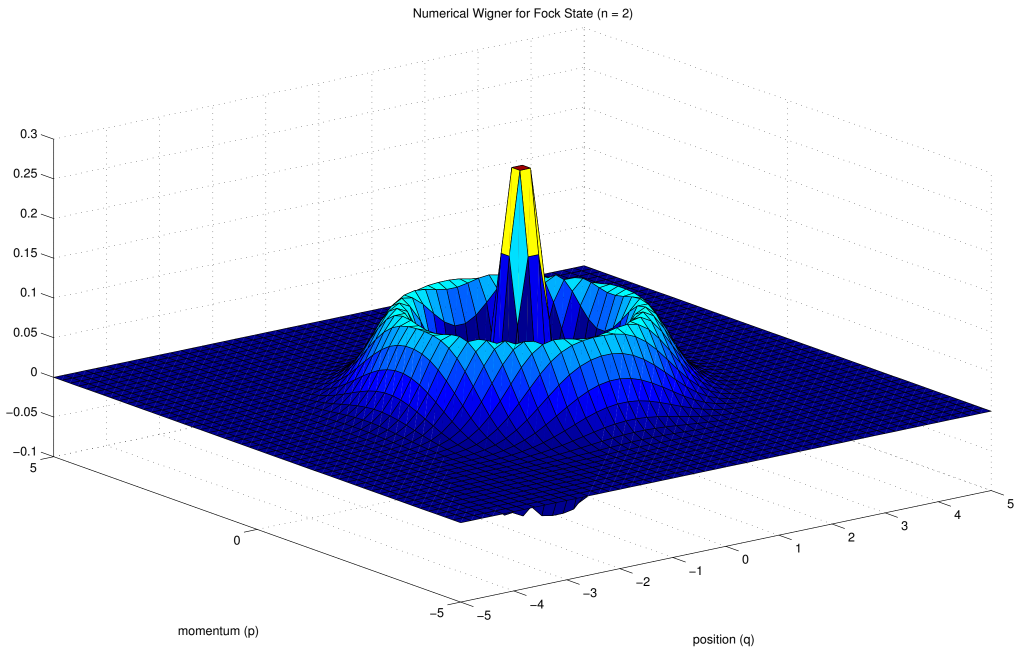

Figure 11: The Fock state for $n = 2$

3.4.4 Numerical Solution $(n = 3)$

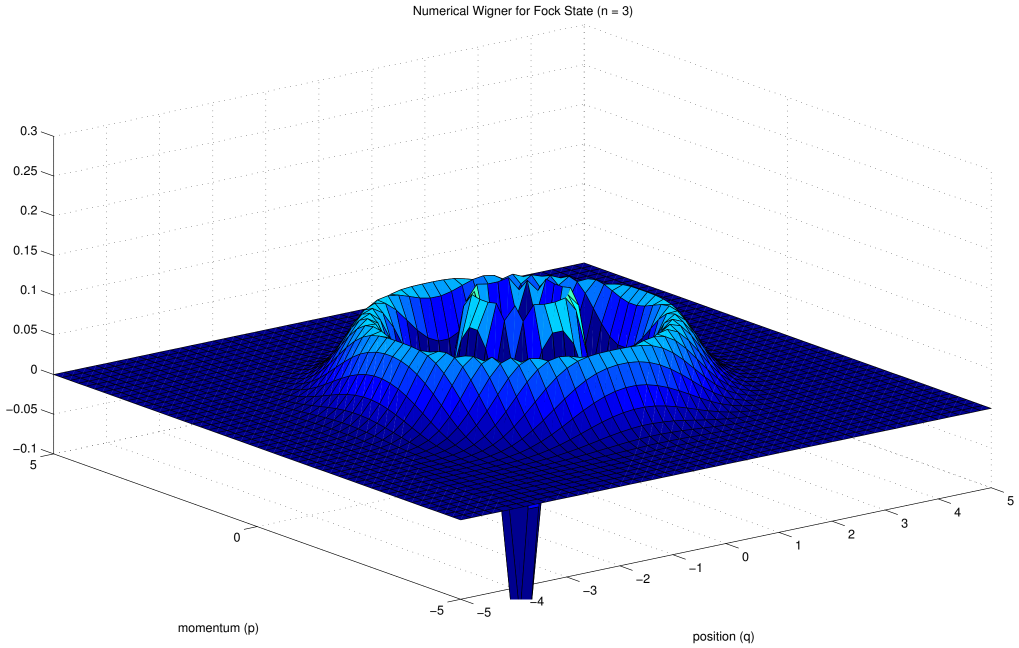

Figure 12: The Fock state for $n = 3$

3.4.5 Numerical Solution $(n = 4)$

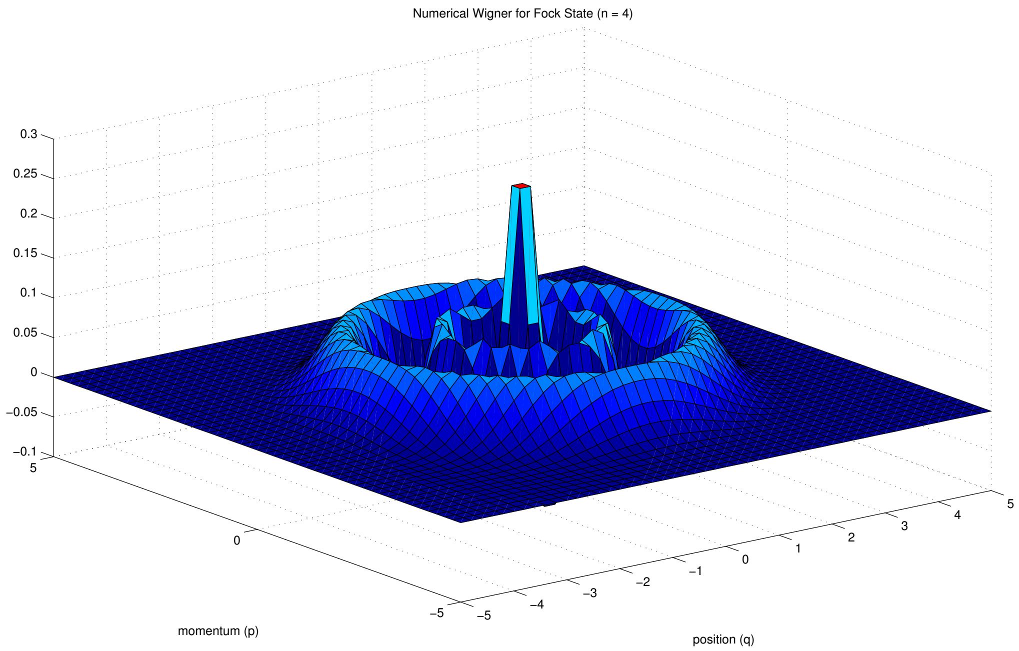

Figure 13: The Fock state for $n = 4$

3.5 Schrödinger Cat States

The Schrodinger cat state plot for $\alpha = (-1 + 2i)$ is shown in Figure 17. The Wigner functions show two coherent states centred around $\pm p = +\sqrt{2}\Im\alpha$, $\pm q = +\sqrt{2}\Re\alpha$, with a mixture of the two pure states, $\mid \alpha\rangle$ and $\mid -\alpha\rangle$, in a wave form in the centre. The value of α defines the position of the two coherent state peaks and the line of symmetry will always be midway between them. Between the two coherent state peaks a wave representing a mixture of both $\mid \alpha\rangle$ and $\mid -\alpha\rangle$ states can be seen. Altering the value of ϕ alters the phase of this wave. There is a noticeable drop in the amplitude of the superposition waves when the separation of the two peaks is rotated into the imaginary direction.

All of the following are numerical solutions

3.5.1 $\alpha = 2.5$, $\phi = 0$

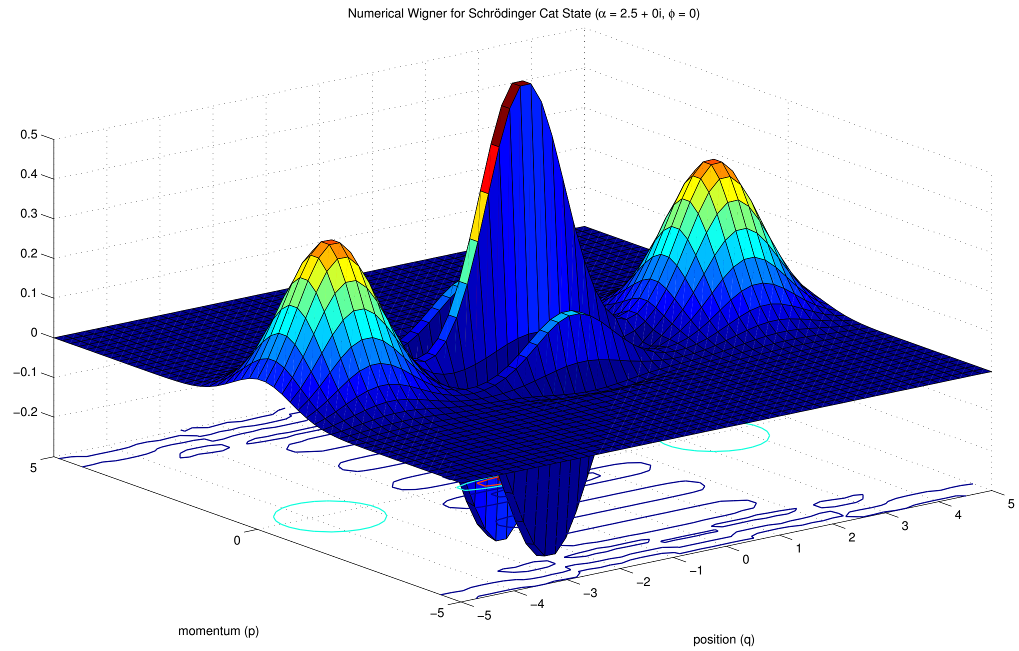

Figure 14: The Cat state for $\alpha = 2.5$ and $\phi = 0$

3.5.2 $\alpha = 2.5$, $\phi = \pi/2$

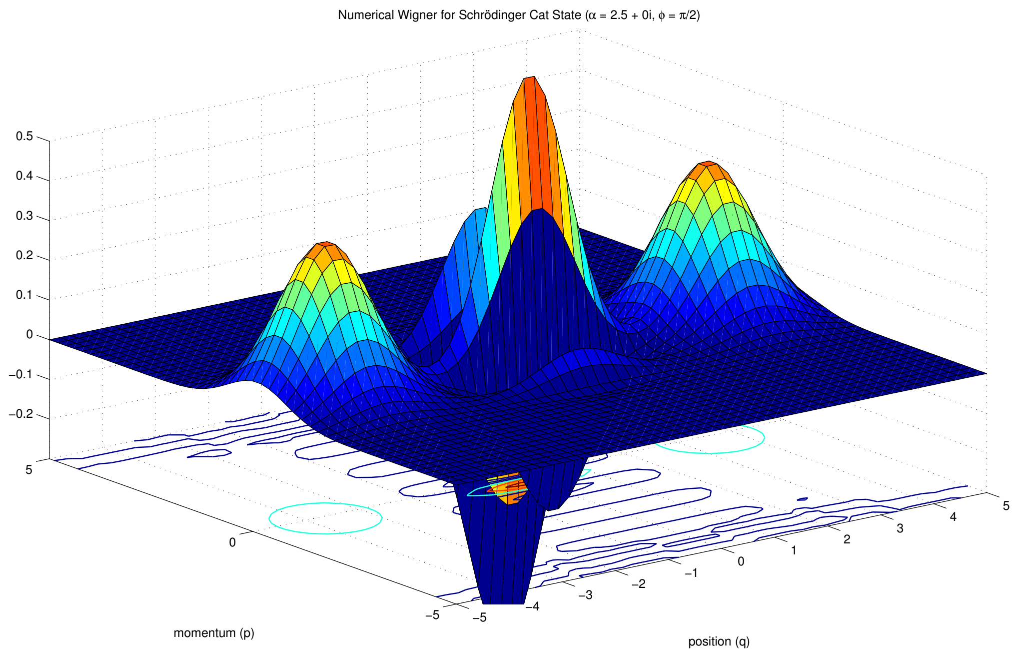

Figure 15: The Cat state for $\alpha = 2.5$ and $\phi = \pi/2$

3.5.3 $\alpha = 2.5$, $\phi = \pi$

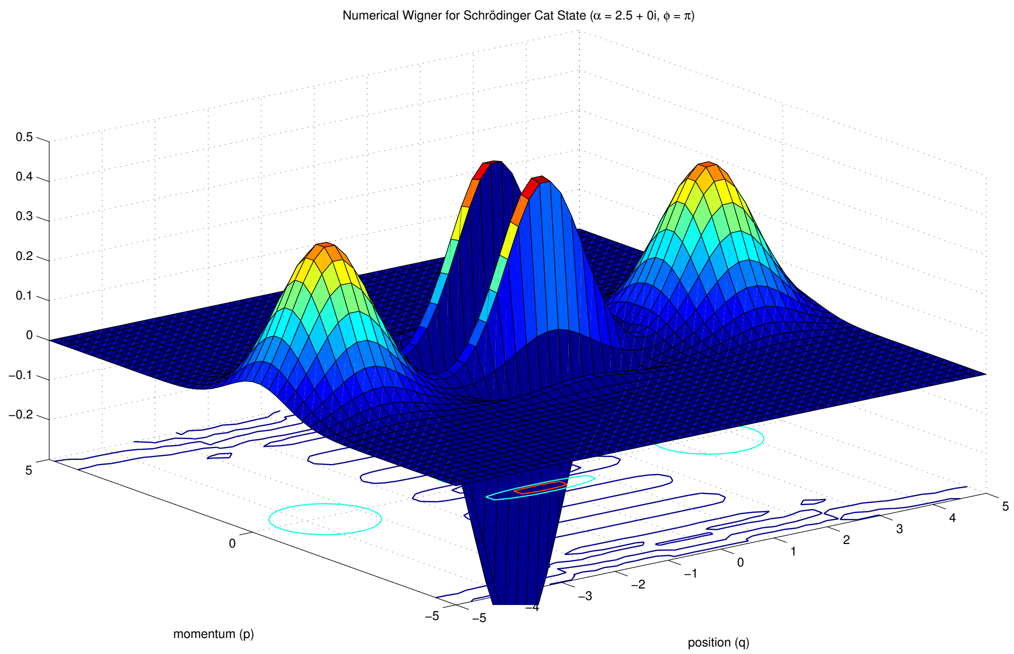

Figure 16: The Cat state for $\alpha = 2.5$ and $\phi = \pi$

3.5.4 $\alpha = -1 + 2i$, $\phi = {}^{\pi}/_{2}$

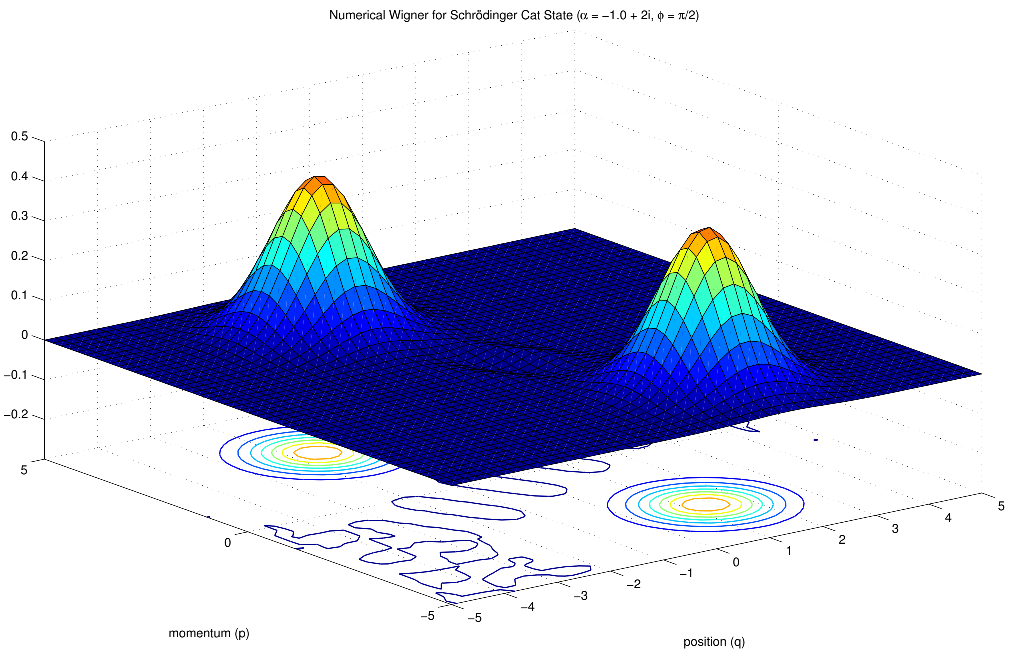

Figure 17: The Cat state for $\alpha = -1 + 2i$ and $\phi = {}^{\pi}/_{2}$

3.6 Mixted States

Figures 18 and 19 show the Wigner function plots for a mixture of two pure coherent states $|\ \alpha\rangle$ and $|\ -\alpha\rangle$ each with an equal probability of being occupied. This differs from the Schrodinger cat state due to the absence of the central combined state wave and just displays two equal Gaussians but with positive and negative values of values of α.

The following are numerical solutions

3.6.1 $\alpha = 1.5,\ P_1 = 0.5,\ P_2 = 0.5$

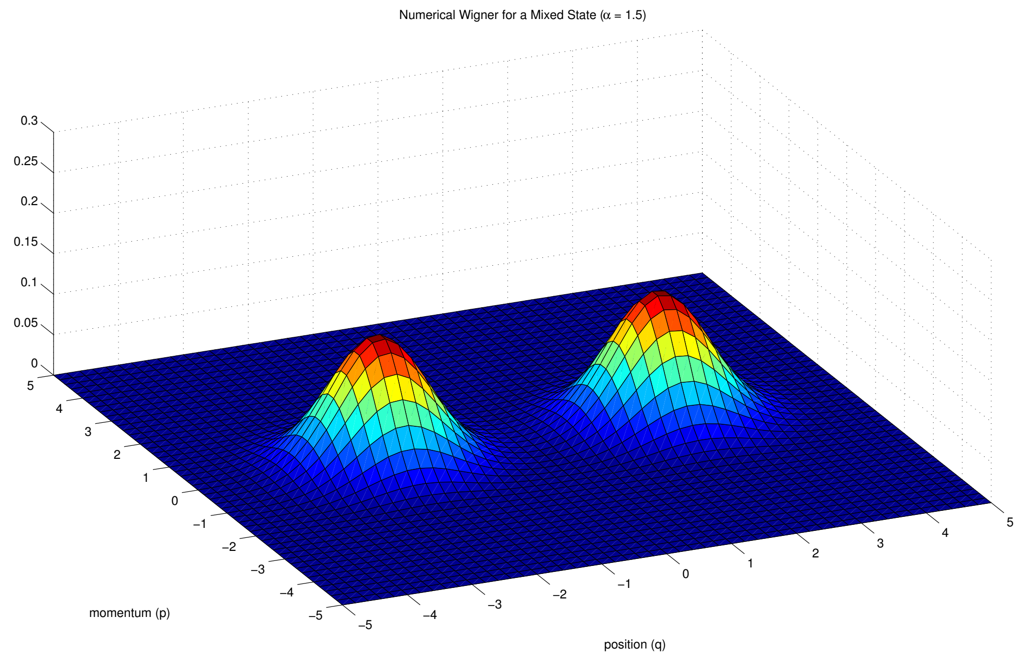

Figure 18: Mixed state at $\alpha = 1.5$

3.6.2 $\alpha = -1 + 2i,\ P_1 = 0.5,\ P_2 = 0.5$

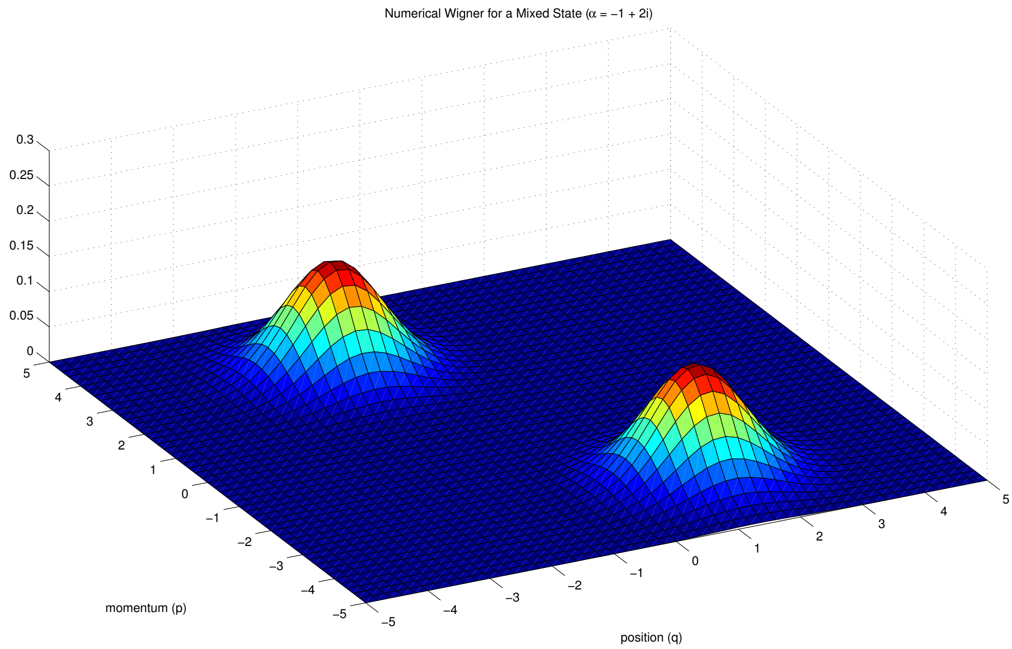

Figure 19: Mixed state at $\alpha = -1 + 2i$

3.7 Thermal States

Plots of the Wigner function for a harmonic oscillator in thermal states at temperatures $T = 3, 5, 7$pK are shown below. The Wigner function is a Gaussian plot that is positive at all points and centred around $p = q = 0$. The standard deviation of the Gaussian is related to the value of T. As T is increased the standard deviation increases and hence the height of the central peak decreases, while the base expands.

3.7.1 $\omega = 1$, $T = 3$pK

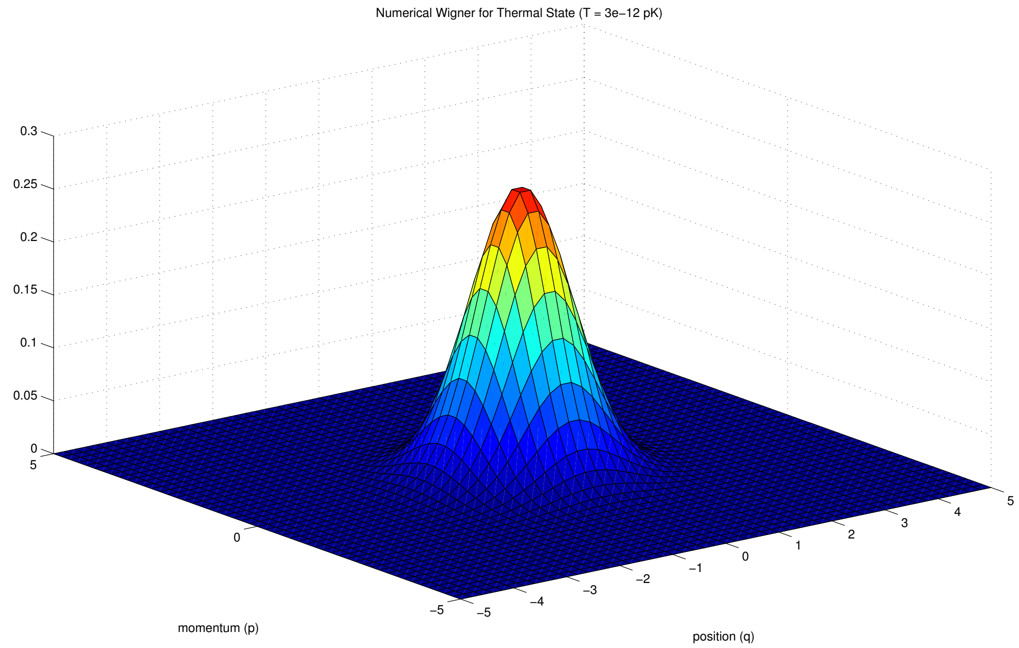

Figure 20: The thermal state for $T = 3$pK

3.7.2 $\omega = 1$, $T = 5$pK

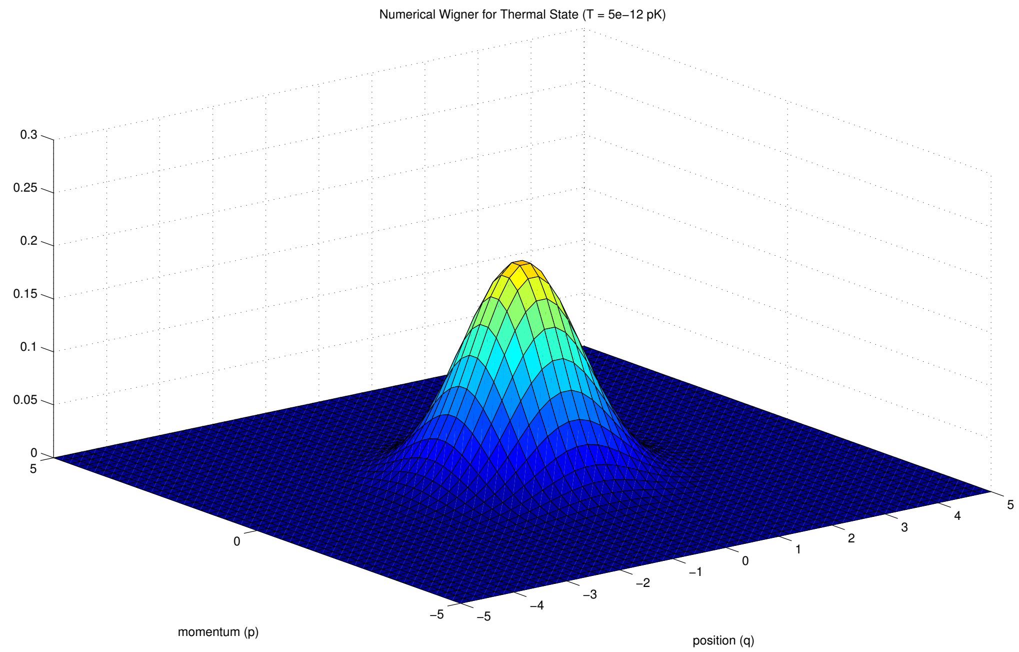

Figure 21: The thermal state for $T = 5$pK

3.7.3 $\omega = 1$, $T = 7$pK

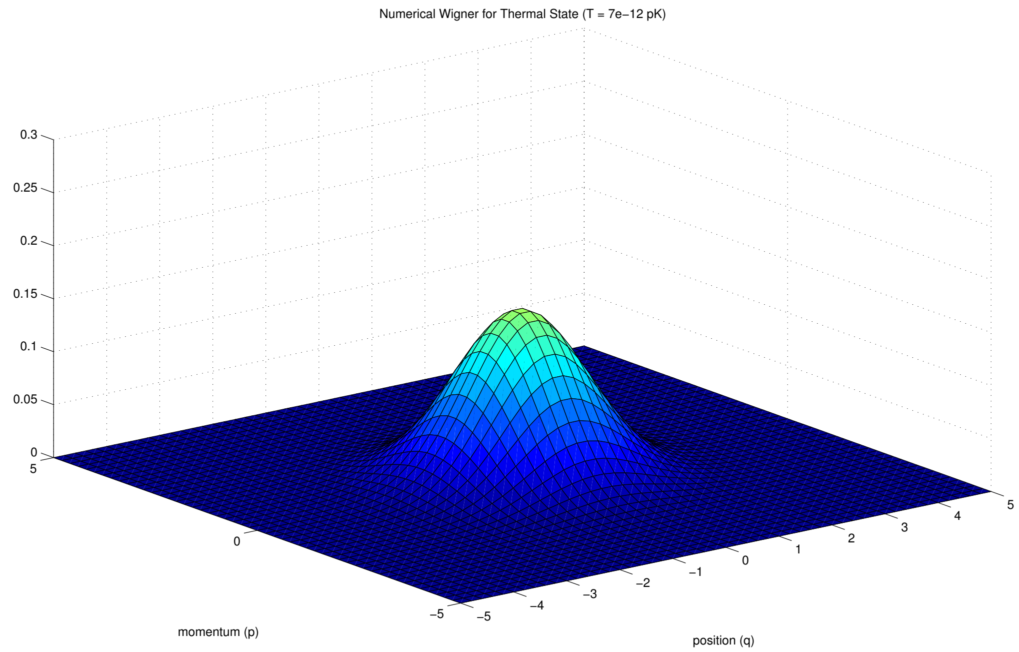

Figure 22: The thermal state for $T = 7$pK

4 Discussion

4.1 Ground State

As this function is always positive it can simply be compared to a classical harmonic oscillator, with initial position $q = 0$ and momentum $p = 0$. It should be noted that these initial conditions are a very special case in the classical world being the only conditions that will cause the oscillator to remain at $x = p = 0$ and not oscillate at all. The Wigner function shows that a quantum oscillator in the ground state will also act in the same way and remain at $x = p = 0$ with no evolution with time. This suggests that the ground state of the quantum harmonic oscillator can be visualised in the same way that classical oscillator is visualised.

4.2 Coherent States

For a purely classical oscillator, for which the position and momentum can be known simultaneously, the probability distribution in phase space would be represented by a delta function centred around the co-ordinates of the oscillator's momentum and position. The standard deviation of the Gaussian in a Wigner function plot can therefore be interpreted as a measure of the uncertainty in the position and momentum of the quantum state due to Heisenberg's uncertainty principle. The low standard deviation values for the plots of the Wigner function for coherent states are therefore a direct consequence of their inherent low uncertainty. Coherent states are the lowest uncertainty quantum states and hence their Wigner function plots have the narrowest Gaussian profiles of any quantum state. This means these states are the closest a quantum state will ever get to a delta function implying that coherent states are the most classical of all quantum states.

4.3 Time Dependant Coherent States

The plots of the evolution of a Wigner function with time, from an initial coherent state, confirm the likeness of coherent states to classical states. A classical harmonic oscillator would move in a circular orbit centred on $p = q = 0$ in exactly the same way a coherent state does. The only difference is again the fact that a coherent state is represented by a Gaussian and not a delta function. Choosing a value of α can be interpreted as setting the initial position and momentum of the oscillator and the subsequent motion of the oscillator can be visualised as we would visualise a classical oscillator.

4.4 Fock/Number States

Number states nicely show how the Wigner function represents a quasi-probability distribution. For $n > 1$ the distribution will always have negative regions, something that could never be seen in a classical probability distribution. Unlike for the coherent state where we would see the same time evolution as in the classical world these number states will remain unchanged in time as they have circular symmetry meaning they have no classical equivalence we can use to visualise them.

4.5 Schrödinger Cat States

Schrödinger cat states, as would be expected, are purely quantum states. The two symmetric Gaussian states represent two coherent states and the central waveform represents a superposition of the two coherent states. It can be thought of as there being an infinite possible number of superposition's of the two states with each superposition being represented by a value of ϕ. The plots show why these states have the name they do as it would be possible not only for a particle to be in either of the coherent states but it may also be in a superposition of both the states at the same time. This can be visualised in the same way as Schrödinger cat, in that it is both alive and dead at the same time.

4.6 Mixture States

Despite the mixed states plot only showing only two Gaussian functions, representing coherent states, and there being no superposition element, it still cannot be considered as a classical state. At no point can it be said in which state a particle would be in until an observation is made, a concept that is not familiar in the classical world. Once a measurement is made on the system it would collapse to being in one of the two coherent states and we could again visualise this classically, but in general this is a purely quantum situation.

4.7 The Thermal State

At temperatures approaching zero Kelvin it can be seen that the Wigner function returns to that of the ground state. This is because at this point the probability of a particle being in a number state of $n > 0$ is very much less than 1 (in this project the cut off probability was set to 10^{-3}) meaning that the oscillator will just be a ground state system. The idea of a Bose-Einstein condensate being produced at ultra cold temperatures is based around this idea for very low values of T, normally in the region of a few hundred nK. Increasing the temperature increases the probability of finding a particle in a number state with $n > 0$. This means that the position and the momentum of the oscillator are less precisely known, hence the standard deviation of the Gaussian increases and the distribution spreads out. In principle the value of T could be raised to a high enough value that the Wigner function would be flat and it would not be possible to know the position or momentum of the particle at all.

5 Conclusion

5.1 Conclusion of Results

From this project we can conclude that coherent states are the lowest uncertainty states, a property that allows them to be classed as the most classical of all quantum states. They can be visualised in the same way a classical oscillator can be visualised but instead of knowing a position and momentum exactly we know them to the lowest possible uncertainty. Coherent states also evolve in time in the same way a classical oscillator would from controllable initial conditions. Number states are conversely very quantum states that do not change in time and have no classical analogy. Schrodinger Cat states and mixed states are completely quantum states. There is no way to tell which state a particle will be in without taking a measurement. Once a measurement has been taken of a particle in the mixed state it will be found to be in one of the two coherent states and can from then on be treated in a very classical way. A measurement of a particle in a Schrodinger cat state could yield the same very classical result but it also offers the highly non-classical option of the particle being measured to be in both states at the same time! A harmonic oscillator in a thermal state at a temperature T has a Gaussian probability in phase space with the standard deviation of the Gaussian controlled by the Temperature. At temperatures of $T < 10^{-12}$K the Wigner function returns to that of the ground state oscillator as the probability of a particle being in a state with $n > 0$ is less than 10^{-3}, but as the temperature increases the probabilities of being in each number state become much closer to each other and so it becomes very hard to say where a particle would be. This leads the quantum harmonic oscillator to be a very high uncertainty state for this model at temperatures $T > 10^{-11}$K.

5.2 Further Study

It would be interesting to continue to study the time dependence of the very quantum states . It would be intriguing to discover how a Schrödinger cat state changed with time and if the probability of being in both states increased or decreased.

A References

[1] William B. Case. Wigner functions and weyl transforms for pedestrians. *American Journal of Physics*, 76(10):937, June 2008.

[2] Christopher C. Gerry and Peter L. Knight. *Introductory Quantum Optics*. Cambridge University Press, 2005.

[3] The MathWorks Inc. Matlab's 'quad' method, November 2009.

[4] Graham Woan. *The Cambridge Handbook of Physics Formulas*. Cambridge University Press, 2003.

B Code

B.1 The Main Program

wigner.m

```
%% Physical Variables
% Coherent States
alpha = -1+2i;

% Number States
n = 4;

% Cat States
phi = pi;

% Thermal States
hbar = 1.055e-34;
K_B = 1.38e-23;
omega = 1;
T = 3e-12;

%% Calculation Variables
resolution = 51;
q_range = [1,-1]*5;
p_range = [1,-1]*5;
integral_limit = 1E4;


%% Program Logic

% For the Thermal State:
A = hbar * omega / (K_B * T);

prob = [];
lastprob = 1; % just to get things going
j = 0;
% Create probabilities for every number state until the probability is ↩
    sufficiently low
while lastprob > 1E-3
    lastprob = exp(-A*j)*(1 - exp(-A));
    prob = [prob lastprob];
    j = j + 1;
end

% For all calculations
W = zeros(resolution,resolution);
q_step = (q_range(1)-q_range(2))/resolution;
p_step = (p_range(1)-p_range(2))/resolution;

qs = linspace(q_range(2),q_range(1),resolution);
```

```
ps = linspace(p_range(2),p_range(1),resolution);

% Create a progress bar to show the progress of the integration
wait = waitbar(0,'Please be patient');
tic

for q_i=1:resolution
    q = q_range(2) + q_i*q_step;

    for p_i=1:resolution
        p = p_range(2) + p_i*p_step;

        % Doing integration over 'all' x
        % N.B. MATLAB takes arguments the wrong way round so p comes before q

          % Only one of the following will be left uncommented so that the required integral is computed

        % Ground State
        W(p_i,q_i) = (2 * pi^1.5)^-1 * quad(@(x)f_psi_0(q,p,x),-integral_limit,integral_limit);

        % Coherent State
        W(p_i,q_i) = (2 * pi^1.5)^-1 * quad(@(x)f_alpha(q,p,x,alpha),-integral_limit,integral_limit);

        % Number States
        W(p_i,q_i) = (2 * pi^1.5)^-1 * quad(@(x)f_psi_n(q,p,x,n),-integral_limit,integral_limit);

        % Schrödinger Cat States
        W(p_i,q_i) = (2 + 2*cos(phi)*exp(-2*alpha*alpha))^-0.5 * ((2*pi)^-1) * quad(@(x)f_sch_cat(q,p,x,alpha,phi),-integral_limit,integral_limit);

        % Mixed State
        W(p_i,q_i) = ((2*pi^1.5)^-1) * quad(@(x)f_mixed(q,p,x,alpha),-integral_limit,integral_limit);

        % Thermal State
        W(p_i,q_i) = (2 * pi^1.5)^-1 * quad(@(x)f_therm(q,p,x,prob),-integral_limit,integral_limit);

        amt = ((q_i-1)*resolution + p_i)/resolution^2;
        if (mod(p_i,6) == 0)
            waitbar(amt,wait,sprintf('Calculating Wigner Function. ETA: %.0fs ',toc/amt * (1 -amt)));
        end
    end
end
close(wait);

%% Visualizing

% The Numerical Solution
subplot(1,2,1); % If needed, the numerical and analytical solutions can be plotted side by side
surf(qs,ps,real(W));
axis([q_range(2),q_range(1),p_range(2),p_range(1),0,0.3]);
title('Numerical Wigner for a particular state');
xlabel('position (q)');
ylabel('momentum (p)');
axis([q_range(2),q_range(1),p_range(2),p_range(1)]);
```

```
% The Analytical Solution
[mesh_qs,mesh_ps] = meshgrid(qs,ps);

% As before, one of the following is left uncommented if an analytical plot is↩
    desired

% Ground State
W = (1/pi) .* exp(- mesh_qs.^2 - mesh_ps.^2);
% Coherent State
W = 1/pi * exp(-(mesh_qs - 2*real(alpha)/sqrt(2)).^2 - (mesh_ps - 2*imag(alpha↩
   )/sqrt(2)).^2);

subplot(1,2,2); % Comment if side—by—side plot not desired
surf(qs,ps,real(W));
title('Analytical Wigner for a particular state');
xlabel('position (q)');
ylabel('momentum (p)');
axis([q_range(2),q_range(1),p_range(2),p_range(1)]);
```

B.2 The Integral Functions

B.2.1 The Ground State

f_psi_0.m

```
% The internals of the intergral of the Wigner function for the ground
% state of the harmonic oscillator
function f = f_psi_0(q,p,x)
f = exp(-(q.^2) - (x.^2)/4) .* exp(1i .* p .* x);
```

B.2.2 The Coherent State

f_alpha.m

```
% The internals of the intergral of the Wigner function for a coherent
% state (alpha) of the Harmonic Oscillator
function f = f_alpha(q,p,x,alpha)
p_J = p - 2*imag(alpha)/sqrt(2);
q_J = q - 2*real(alpha)/sqrt(2);
f = exp(-q_J.^2 - (x.^2)/4 - 1i*p_J*x);
```

B.2.3 The Time Dependent Coherent State

wigner_time.m

```
% Special case: Time Dependant Coherent States
%% Physical Constants
m = 1;
w = 1;

%% Calculation Variables
resolution = 51;
q_range = [1,-1]*5;
p_range = [1,-1]*5;
[mesh_qs,mesh_ps] = meshgrid(qs,ps);

%% Plotting
n = 1;
for t = linspace(0,pi/2,4)
```

```
    W = 1/pi * exp(-(mesh_qs * cos(w * t) - mesh_ps * sin(w*t) - 2*real(alpha)↩
        /sqrt(2)).^2 - (mesh_qs * sin(w * t) + mesh_ps * cos(w*t) - 2*imag(↩
        alpha)/sqrt(2)).^2);
    subplot(2,2,n);
    surf(qs,ps,real(W));
    title(sprintf('Analytical Wigner for the Coherent State (\\alpha = 1 - i, ↩
        t = %.0f/4 * \\pi/2)',n));
    xlabel('position (q)');
    ylabel('momentum (p)');
    axis([q_range(2),q_range(1),p_range(2),p_range(1),min(min(W)),max(max(W))↩
        ]);
    pause(0.2);
    n = n + 1;
end
```

B.2.4 The Fock States

f_psi_n.m

```
% The internals of the intergral of the Wigner function for the Number
% states for the Harmonic Oscialltor
function f = f_psi_n(q,p,x,n)
f = exp(1i * p * x)/(factorial(n) * 2^n) .* Hermite(q - x/2,n) .* Hermite(q + ↩
    x/2,n) .* exp(-q.^2 - x.^2/4);
```

B.2.5 The Schrödinger Cat State

f_sch_cat.m

```
% The internals of the intergral of the Wigner function for a Schrödinger
% Cat state of a Harmonic Oscillator
function f = f_sch_cat(q,p,x,alpha,phi)
p_J = p - 2*imag(alpha)/sqrt(2);
q_J = q - 2*real(alpha)/sqrt(2);
p_K = p + 2*imag(alpha)/sqrt(2);
q_K = q + 2*real(alpha)/sqrt(2);

f = exp(-q_J^2 - (x.^2)/4 - 1i*p_J*x) + ... % One coherent State
    exp(-q_K.^2 - (x.^2)/4 - 1i*p_K*x) + ... % Opposide Coherent State
    exp( 1i * p * x) .* ( ...
        exp(+ 1i*phi - alpha * conj(alpha) - q^2 - x.^2/4 - 2i * q * imag(↩
            alpha) + x * real(alpha)) + ...
        exp(- 1i*phi - alpha * conj(alpha) - q^2 - x.^2/4 + 2i * q * imag(↩
            alpha) - x * real(alpha)) ...
    );
```

B.2.6 The Mixed State

f_mixed.m

```
% The internals of the intergral of the Wigner function for a mixed
% state (alpha) of the Harmonic Oscillator
function f = f_mixed(q,p,x,alpha)
p_J = p - 2*imag(alpha)/sqrt(2);
q_J = q - 2*real(alpha)/sqrt(2);
p_K = p + 2*imag(alpha)/sqrt(2);
q_K = q + 2*real(alpha)/sqrt(2);

f = 0.5 * exp(-q_J.^2 - (x.^2)/4 - 1i*p_J*x) + 0.5 * exp(-q_K.^2 - (x.^2)/4 - ↩
    1i*p_K*x);
```

B.2.7 The Thermal State

f_therm.m

```
% The internals of the intergral of the Wigner function for the Thermal
% state of the Harmonic Oscialltor
function f = f_therm(q,p,x,probs)
n = 0;
f = 0;
for prob=probs
    f = f + prob * exp(1i * p * x)/(factorial(n) * 2^n) .* Hermite(q - x/2,n) ↩
        .* Hermite(q + x/2,n) .* exp(-q.^2 - x.^2/4);
    n = n + 1;
end
```

B.3 Calculating Hermite Polynomials

Hermite.m

```
% Calculates the (Physicist's) Hermite Polynomial
function H = Hermite(y,n)
switch (n)
    % Some pre-calculated hermite polynomials to make things speedier
    case 0
        H = 1;
    case 1
        H = 2 * y;
    case 2
        H = 4 * y.^2 - 2;
    case 3
        H = 8 * y.^3 - 12*y;
    otherwise
        H = 2*y.*Hermite(y,n-1) - 2*(n-1)*Hermite(y,n-2);
end
```

C Derrivations

C.1 Ground State Wigner Function

C.1.1 For the Numerical Calculation

Using the harmonic oscillator's ground state wavefunction from (eq.6) and the integral form of the Wigner function (eq.3):

$$\begin{aligned} W(q,p) &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\psi^*\left(q-\frac{x}{2}\right)\cdot\psi\left(q+\frac{x}{2}\right)e^{(-ipx/\hbar)}dx \\ &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\left(\frac{1}{a\sqrt{\pi}}\right)^{1/2}\exp\left(\frac{\left(q-\frac{x}{2}\right)^2}{2a^2}\right)\cdot\left(\frac{1}{a\sqrt{\pi}}\right)^{1/2}\exp\left(\frac{\left(q+\frac{x}{2}\right)^2}{2a^2}\right)e^{-ipx/\hbar}dx \\ &= \frac{1}{2\pi\hbar a\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\left(-\left(q^2+\frac{x^2}{4}-qx\right)-\left(q^2+\frac{x^2}{4}+qx\right)\right)e^{-ipx/2a^2\hbar}dx \end{aligned}$$

At this point, as explained in section 2.1.1, the position and momentum variables are replaced, altering the length and mass dimensions like so:

$$q_0=\tfrac{q}{a};\ \left[x_0=\tfrac{x}{a};\ \tfrac{dx}{dx_0}=a\right]\ p_0=\tfrac{pa}{\hbar}$$

$$\begin{aligned} W(q_0,p_0) &= \frac{1}{2\pi\hbar a\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\left(-q_0^2-\frac{x_0^2}{4}\right)e^{-ip_0x_0}dx_0\cdot a \\ &= \frac{1}{2\pi\sqrt{\pi}\hbar}\int_{-\infty}^{\infty}\exp\left(-q_0^2-\frac{x_0^2}{4}-ip_0x_0\right)dx_0 \end{aligned}$$

But $W(q_0,p_0)$ does not have the correct dimensions for the changes we made above. Because we know that the integral of the Wigner function over all phase space must be 1 – it is a probability – we can state that

$$\iint_{-\infty}^{\infty}W(q_0,p_0)\,dq_0dp_0 = 1$$

And thus that

$$\begin{aligned} \frac{q_0}{q}\frac{p_0}{p}W_0(q_0,p_0) &= W(q_0,p_0) \\ \frac{1}{\hbar}W_0(q_0,p_0) &= W(q_0,p_0) \end{aligned}$$

And so the final (numerically solvable) Wigner equation for the ground state of the harmonic oscillator is found as:

$$W_0(q_0,p_0)=\frac{1}{2\pi\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\left(-q_0^2-\frac{x_0^2}{4}-ip_0x_0\right)dx_0 \tag{13}$$

C.1.2 The Analytical Solution

Taking the inside of the exponential in the numerical equation (eq. 2) and completing the square

$$\begin{aligned} -\frac{x_0^2}{4}-ip_0x_0-q_0^2 &= -(Ax+B)^2-C-q_0^2 \\ &= -\left(\frac{x_0}{2}+ip_0\right)^2+p_0^2-q_0^2 \end{aligned}$$

Letting $y=\left(\frac{x_0}{2}+2ip_0\right)$, and thus $\frac{dy}{dx_0}=2$ then the Wigner equation becomes an analytically solvable integral:

$$\begin{aligned} W_0(q_0,p_0) &= \frac{1}{2\pi\sqrt{\pi}}\exp\left(p_0^2-q_0^2\right)\int_{-\infty}^{\infty}e^{-y^2}dy\cdot 2 \\ &= \frac{1}{\pi\sqrt{\pi}}\exp\left(p_0^2-q_0^2\right)\sqrt{\pi} \end{aligned}$$

Giving the final solution for the Wigner function of the Ground State Harmonic Oscillator:

$$W_0(q_0,p_0)=\frac{1}{\pi}\exp\left(p_0^2-q_0^2\right) \tag{14}$$

C.2 Coherent States

C.2.1 For Numerical Calculation

Using the harmonic oscialltor's coherent state wavefunction from (eq.7) as well as the definitions that:

$$p_n=2\Re\{\alpha\};\ x_n=2\Im\{\alpha\}\ \Rightarrow\ \alpha=\tfrac{1}{2}(x_n+ip_n)$$

The wavefunctions can be refactored to become:

$$\begin{aligned} \psi_\alpha(q) &= \left(\frac{1}{a\sqrt{\pi}}\right)^{1/2}e^{ix_np_n/4}e^{-p_n^2/4}\exp\left(-\frac{1}{2}\cdot\left(\frac{q}{a}-\frac{-x_n}{\sqrt{2}}+\frac{ip_n}{\sqrt{2}}\right)^2\right) \\ \psi_\alpha^*\left(q\text{-}\frac{x}{2}\right)\psi_\alpha\left(q\text{+}\frac{x}{2}\right) &= \frac{1}{a\sqrt{\pi}}e^{-p_n^2/2}\exp\left(-\frac{1}{2}\cdot\left(\frac{q}{a}-\frac{-x_n}{\sqrt{2}}+\frac{ip_n}{\sqrt{2}}\right)^2\right)\exp\left(-\frac{1}{2}\cdot\left(\frac{q}{a}-\frac{-x_n}{\sqrt{2}}-\frac{ip_n}{\sqrt{2}}\right)^2\right) \\ &= \frac{1}{a\sqrt{\pi}}\exp\left(-q_1^2-\frac{x^2}{4a^2}+\frac{ixp_n}{a\sqrt{2}}\right) \end{aligned}$$

Where (making the same dimensional substitutions as before):

$$q_1=\tfrac{q}{a}-\tfrac{x_n}{\sqrt{2}a}\cdot a=\tfrac{q}{a}-\sqrt{2}\Re\{\alpha\};\ p_1=\tfrac{p}{a}-\tfrac{p_n\hbar}{a\sqrt{2}}\cdot\tfrac{a}{\hbar}=\tfrac{p}{a}-\sqrt{2}\Im\{\alpha\};\ x_0=\tfrac{x}{a}$$

Continuing using the integral form of the Wigner function (eq.3):

$$\begin{aligned} W(q,p) &= \frac{1}{2\pi a\sqrt{\pi}\hbar}\int_{-\infty}^{\infty}\psi^*\left(q-\frac{x}{2}\right)\cdot\psi\left(q+\frac{x}{2}\right)e^{(-ipx/\hbar)}dx \\ W(q_0,p_0) &= \frac{1}{2a\pi\sqrt{\pi}\hbar}\int_{-\infty}^{\infty}\exp\left(-q_1^2-\frac{x_0^2}{4}-ix_0p_1\right)dx_0\cdot a \\ W_0(q_0,p_0) &= \frac{1}{2\pi\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\left(-q_1^2-\frac{x_0^2}{4}-ix_0p_1\right)dx_0 \end{aligned}$$

It is clearly visible that this has the same form as for the Ground state, except that

$$\begin{aligned} q_1 &= q_0-\sqrt{2}\Re\{\alpha\} \\ p_1 &= p_0-\sqrt{2}\Im\{\alpha\} \end{aligned}$$

Hereby proving that the wigner function of the coherent state is simply a transposition of the wigner function of the ground state in phase-space according to the value of α.

C.2.2 The Analytical Solution

In the same fashion as with the Ground state solution above it can be found that:

$$W_0(q_0,p_0)=\frac{1}{\pi}\exp\left(p_1^2-q_1^2\right)=\frac{1}{\pi}\exp\left(\left(p_0-\sqrt{2}\Im\{\alpha\}\right)^2-\left(q_0-\sqrt{2}\Re\{\alpha\}\right)^2\right) \tag{15}$$

C.3 Schrödinger Cat States

Schrödinger Cat states have a wavefunction:

$$
\begin{aligned}
\mid \psi\left(\phi\right)\rangle &= N\left(\phi\right)\left(\mid \alpha\rangle + e^{i\phi}\mid -\alpha\rangle\right) \\
N\left(\phi\right) &= \left(2 + 2\cos\left(\phi\right)e^{-2\alpha^{2}}\right)^{-1/2}
\end{aligned}
$$

Using the Dirac notation form of the Wigner function

$$
W\left(q,p\right) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\langle q-\frac{x}{2}\mid \psi\left(\phi\right)\rangle\langle\psi\left(\phi\right)\mid q+\frac{x}{2}\rangle e^{ipx/\hbar}dx
$$

It can be seen that replacing the $\psi\left(\phi\right)$ with the Cat wavefunction will create four terms. Two of them will be the coherent states of α and $-\alpha$ while the remaining two will be superpositions of α and $-\alpha$. Using the following coherent state identity these can be solved.

$$
\langle\alpha\mid\beta\rangle = \exp\left(-\frac{1}{2}\cdot\left(\left|\alpha\right|^{2}+\left|\beta\right|^{2}-2\alpha^{*}\beta\right)\right)
$$